# ROOTS OF UNITY AND ADDITIVE REPRESENTATION FUNCTIONS

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**Abstract**

*Let* $\mathrm{A}$ *be an infinite, strictly increasing sequence of non-negative integers, and for* $n \in \square$ *, let*

$$R^*(n) = \left|\left\{(k;l) \mid a_k + a_l = n,\ a_k \in \mathrm{A},\ a_l \in \mathrm{A},\ a_k \le a_l\right\}\right|.$$

*It is known (Horváth [2007]) that* $1 \le R^*(n) \le 2$ *cannot hold from a point on. We will prove this result by using roots of unity, without integral.*


**Keywords**: *root of unity, sequence, additive representation function.*

## Egységgyökök és additív reprezentáció függvények


**Összefoglalás**

*Legyen* $\mathrm{A}$ *nemnegatív egészeknek egy végtelen, szigorúan növekedő sorozata, és* $n \in \square$ *esetén legyen*

$$R^*(n) = \left|\left\{(k;l) \mid a_k + a_l = n,\ a_k \in \mathrm{A},\ a_l \in \mathrm{A},\ a_k \le a_l\right\}\right|.$$

*Ismert (Horváth [2007]), hogy nem lehet valahonnantól kezdve* $1 \le R^*(n) \le 2$. *Ezt az eredményt egységgyökök használatával, integrál nélkül fogjuk bizonyítani.*


**Kulcsszavak:** *egységgyök, sorozat, additív reprezentáció-függvény.*

## 1. Introduction

If $n$ is a positive integer, then a complex number $z$ is called an $n$ th root of unity if $z^n = 1$. A complex number is called a root of unity if it is an $n$ th root of unity for some positive integer $n$. Let $M$ be an arbitrary positive integer, and for $k \in \square$ , let

$$e\left(\frac{k}{M}\right) = e^{\frac{2\pi i k}{M}} \quad ,(1)$$

which is an $M$ th root of unity.

Furthermore, let $\mathrm{A}$ be an infinite, strictly increasing sequence of non-negative integers, and for $n \in \square$ , let

$$R^*(n) = \left|\left\{(k;l) \mid a_k + a_l = n,\ a_k \in \mathrm{A},\ a_l \in \mathrm{A},\ a_k \le a_l\right\}\right|, \qquad (2)$$

$$R(n) = \left|\left\{(k;l) \mid a_k + a_l = n,\ a_k \in \mathrm{A},\ a_l \in \mathrm{A}\right\}\right| \qquad ,(3)$$

which are called representation functions of the sequence $\mathrm{A}$ . It is shown (Horváth [2007]), that if $d > 0$ is an integer, then

$$d \le R^*(n) \le d + \left[ \sqrt{2d} + \frac{1}{2} \right]$$

cannot hold for $n > n_0$. For $d = 1$, we get the following statement:

THEOREM. *It is impossible that*

$$1 \le R^*(n) \le 2 \ \text{for} \ n \ge n_0 \cdot (4)$$

The proof of this result in [1] contains integrals, but now we will prove it by sums involving roots of unity instead of integrals.

## 2. A lemma

LEMMA. *If* $l \in □$ *, then*

$$\sum_{k=0}^{M-1} e\left(\frac{kl}{M}\right) = \begin{cases} 0 & if \ \ M \ \ is \ not \ a \ divisor \ of \ \ l, \\ M & if \ \ M \mid l. \end{cases}$$

Proof of the lemma. If $M$ is a divisor of $l$, then every term in the sum is equal to 1. If $M$ is not a divisor of $l$, then by (1),

$$\sum_{k=0}^{M-1} e\left(\frac{kl}{M}\right) = \sum_{k=0}^{M-1} e^{\frac{2\pi ikl}{M}} = \sum_{k=0}^{M-1} \left( e^{\frac{2\pi il}{M}} \right)^k = \frac{1 - \left( e^{\frac{2\pi il}{M}} \right)^M}{1 - e^{\frac{2\pi il}{M}}} = \frac{1 - e^{2\pi il}}{1 - e^{\frac{2\pi il}{M}}} = 0.$$

## 3. Proof of the theorem

By indirect argument, let us suppose that (4) holds for some positive integer $n_0$. For $|z| < 1$, let

$$F(z) = \sum_{a \in \mathrm{A}} z^a \tag{5}$$

be the generating function of the sequence $\mathrm{A}$, then by (3),

$$F^2(z) = \left( \sum_{a_k \in \mathrm{A}} z^{a_k} \right) \left( \sum_{a_l \in \mathrm{A}} z^{a_l} \right) = \sum_{n=0}^{\infty} R(n) z^n. \tag{6}$$

Let $M$ and $N$ be "large" positive integers, and let

$$r = 1 - \frac{1}{N}. \tag{7}$$

For $\lambda \in □$, let us consider the inequality

$$\left(\left|F^2(z)-\lambda\frac{1}{1-z}\right|-F\left(r^2\right)\right)^2\geq 0. \qquad (8)$$

Setting $z=re\left(\frac{k}{M}\right)$ in (8) for $k=0,\ 1,\ \ldots,\ M-1,$ and adding these expressions, (in view of (5) and (8)) we have

$$\sum_{k=0}^{M-1}\left(\left|\left(\sum_{a\in A}r^a e\left(\frac{ka}{M}\right)\right)^2-\lambda\frac{1}{1-re\left(\frac{k}{M}\right)}\right|-\sum_{a\in A}r^{2a}\right)^2\geq 0. \qquad (9)$$

By appropriating choice of $M$ , $N$ and $\lambda$ , we will deduce a contradiction from (9). Taking the square, (9) can be written in the form

$$\sum_{k=0}^{M-1}\left|\left(\sum_{a\in A}r^a e\left(\frac{ka}{M}\right)\right)^2-\lambda\frac{1}{1-re\left(\frac{k}{M}\right)}\right|^2-2\left(\sum_{a\in A}r^{2a}\right)\sum_{k=0}^{M-1}\left|\left(\sum_{a\in A}r^a e\left(\frac{ka}{M}\right)\right)^2-\lambda\frac{1}{1-re\left(\frac{k}{M}\right)}\right|$$

$$+\sum_{k=0}^{M-1}\left(\sum_{a\in A}r^{2a}\right)^2\geq 0. \qquad (10)$$

Since

$$\frac{1}{1-re\left(\frac{k}{M}\right)}=\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n=\sum_{n=0}^{\infty}r^n e\left(\frac{kn}{M}\right),$$

therefore by (6) and (10),

$$\sum_{k=0}^{M-1}\left|\sum_{n=0}^{\infty}\left(R(n)-\lambda\right)r^n e\left(\frac{kn}{M}\right)\right|^2-2\left(\sum_{a\in A}r^{2a}\right)\sum_{k=0}^{M-1}\left|\left(\sum_{a\in A}r^a e\left(\frac{ka}{M}\right)\right)^2-\lambda\frac{1}{1-re\left(\frac{k}{M}\right)}\right|$$

$$+M\sum_{n=0}^{\infty}R(n)r^{2n}\geq 0,$$

that is, by the triangle inequality,

$$\sum_{k=0}^{M-1}\left(\sum_{n=0}^{\infty}\left(R(n)-\lambda\right)r^n e\left(\frac{kn}{M}\right)\right)\left(\sum_{n'=0}^{\infty}\left(R(n')-\lambda\right)r^{n'}e\left(-\frac{kn'}{M}\right)\right)+M\sum_{n=0}^{\infty}R(n)r^{2n}$$

$$\geq 2\left(\sum_{a\in A}r^{2a}\right)\left(\sum_{k=0}^{M-1}\left|\left(\sum_{a\in A}r^a e\left(\frac{ka}{M}\right)\right)^2\right|-\sum_{k=0}^{M-1}\left|\lambda\frac{1}{1-re\left(\frac{k}{M}\right)}\right|\right)$$

$$=2\left(\sum_{a\in A} r^{2a}\right)\sum_{k=0}^{M-1}\left(\sum_{a\in A} r^{a} e\left(\frac{ka}{M}\right)\right)\left(\sum_{a'\in A} r^{a'} e\left(-\frac{ka'}{M}\right)\right)-2\left(\sum_{a\in A} r^{2a}\right)|\lambda|\sum_{k=0}^{M-1}\frac{1}{\left|1-re\left(\frac{k}{M}\right)\right|}$$

. (11)

By changing the order of summations,

$$\sum_{k=0}^{M-1}\left(\sum_{a\in A} r^{a} e\left(\frac{ka}{M}\right)\right)\left(\sum_{a'\in A} r^{a'} e\left(-\frac{ka'}{M}\right)\right)=\sum_{a\in A}\sum_{a'\in A} r^{a+a'}\sum_{k=0}^{M-1} e\left(\frac{k(a-a')}{M}\right), \quad (12)$$

where the most inner sum, by the lemma, is $0$ or $M$, thus

$$\sum_{a\in A}\sum_{a'\in A} r^{a+a'}\sum_{k=0}^{M-1} e\left(\frac{k(a-a')}{M}\right)\geq M\sum_{a\in A} r^{2a}. \quad (13)$$

Furthermore (applying the lemma),

$$\sum_{k=0}^{M-1}\left(\sum_{n=0}^{\infty}(R(n)-\lambda)r^{n} e\left(\frac{kn}{M}\right)\right)\left(\sum_{n'=0}^{\infty}(R(n')-\lambda)r^{n'} e\left(-\frac{kn'}{M}\right)\right)$$

$$=\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}(R(n)-\lambda)(R(n')-\lambda)r^{n+n'}\sum_{k=0}^{M-1} e\left(\frac{k(n-n')}{M}\right)$$

$$=M\sum_{n=0}^{\infty}(R(n)-\lambda)^{2} r^{2n}+M\sum_{n=0}^{\infty}\sum_{\substack{n'=0\\ M|(n-n')\\ n\neq n'}}^{\infty}(R(n)-\lambda)(R(n')-\lambda)r^{n+n'}. \quad (14)$$

By the indirect assumption, there exists a positive number $c$ that $R^{*}(n)\leq c$, so $R(n)\leq 2R^{*}(n)\leq 2c$, therefore

$$M\sum_{n=0}^{\infty}\sum_{\substack{n'=0\\ M|(n-n')\\ n\neq n'}}^{\infty}(R(n)-\lambda)(R(n')-\lambda)r^{n+n'}\leq M(2c+|\lambda|)^{2}\sum_{n=0}^{\infty}\sum_{\substack{n'=0\\ M|(n-n')\\ n\neq n'}}^{\infty} r^{n+n'}$$

$$=2M(2c+|\lambda|)^{2}\sum_{n=0}^{\infty}\sum_{t=1}^{\infty} r^{2n+tM}=2M(2c+|\lambda|)^{2}\sum_{n=0}^{\infty} r^{2n}\sum_{t=1}^{\infty}\left(r^{M}\right)^{t}$$

$$=2M(2c+|\lambda|)^{2}\frac{1}{1-r^{2}}r^{M}\frac{1}{1-r^{M}}. \quad (15)$$

Thus by (11)-(15),

$$M\sum_{n=0}^{\infty}(R(n)-\lambda)^{2} r^{2n}+2M(2c+|\lambda|)^{2}\frac{1}{1-r^{2}}\frac{r^{M}}{1-r^{M}}+M\sum_{n=0}^{\infty}R(n)r^{2n}$$

$$\geq 2M\left(\sum_{a\in A} r^{2a}\right)^{2}-2\left(\sum_{a\in A} r^{2a}\right)|\lambda|\sum_{k=0}^{M-1}\frac{1}{\left|1-re\left(\frac{k}{M}\right)\right|}. \quad (16)$$

Since

$$\left|1-re\left(\frac{k}{M}\right)\right|^2=\left(1-r\cos\frac{2\pi k}{M}\right)^2+r^2\left(\sin\frac{2\pi k}{M}\right)^2$$

$$=(1-r)^2+2r\left(1-\cos\frac{2\pi k}{M}\right)=(1-r)^2+4r\left(\sin\frac{\pi k}{M}\right)^2$$

and $\sin\frac{\pi k}{M}\geq\frac{2}{\pi}\frac{\pi k}{M}=\frac{2k}{M}$ for $0\leq\frac{\pi k}{M}\leq\frac{\pi}{2}$, therefore

$$\sum_{k=0}^{M-1}\frac{1}{\left|1-re\left(\frac{k}{M}\right)\right|}\leq\sum_{k=-\left[\frac{M}{2}\right]}^{\left[\frac{M}{2}\right]}\frac{1}{\left|1-re\left(\frac{k}{M}\right)\right|}\leq 2\left(\sum_{k=0}^{\left[\frac{M}{N}\right]}\frac{1}{\left|1-re\left(\frac{k}{M}\right)\right|}+\sum_{k=\left[\frac{M}{N}\right]+1}^{\left[\frac{M}{2}\right]}\frac{1}{\left|1-re\left(\frac{k}{M}\right)\right|}\right)$$

$$\leq 2\left(\sum_{k=0}^{\left[\frac{M}{N}\right]}\frac{1}{1-r}+\sum_{k=\left[\frac{M}{N}\right]+1}^{\left[\frac{M}{2}\right]}\frac{1}{2\sqrt{r}\left|\sin\frac{\pi k}{M}\right|}\right)\leq 2\left(\frac{1}{1-r}\left(\left[\frac{M}{N}\right]+1\right)+\frac{1}{2\sqrt{r}}\sum_{k=\left[\frac{M}{N}\right]+1}^{\left[\frac{M}{2}\right]}\frac{M}{2k}\right)$$

$$=2\left(\frac{1}{1-r}\left(\left[\frac{M}{N}\right]+1\right)+\frac{M}{4\sqrt{r}}\sum_{k=\left[\frac{M}{N}\right]+1}^{\left[\frac{M}{2}\right]}\frac{1}{k}\right). \tag{17}$$

Furthermore, since $e<\left(1+\frac{1}{n}\right)^{n+1}$ for $n=1,\ 2,\ \ldots,$ and $[x]>\frac{x}{2}$ for $x\geq 1$, so if $M\geq N\geq 2$, then

$$\sum_{k=\left[\frac{M}{N}\right]+1}^{\left[\frac{M}{2}\right]}\frac{1}{k}=\sum_{k=\left[\frac{M}{N}\right]+1}^{\left[\frac{M}{2}\right]}\frac{\ln e}{k}\leq\sum_{k=\left[\frac{M}{N}\right]+1}^{\left[\frac{M}{2}\right]}\frac{\ln\left(\left(1+\frac{1}{k-1}\right)^k\right)}{k}=\sum_{k=\left[\frac{M}{N}\right]+1}^{\left[\frac{M}{2}\right]}\ln\left(1+\frac{1}{k-1}\right)$$

$$=\sum_{k=\left[\frac{M}{N}\right]+1}^{\left[\frac{M}{2}\right]}\left(\ln k-\ln(k-1)\right)=\ln\left[\frac{M}{2}\right]-\ln\left[\frac{M}{N}\right]\leq\ln\frac{M}{2}-\ln\frac{M}{2N}=\ln N;$$

and by (7), $r\geq\frac{1}{2}$ for $N\geq 2$, thus in view of (7) and (17) for $M\geq N\geq 2$,

$$\sum_{k=0}^{M-1}\frac{1}{\left|1-re\left(\frac{k}{M}\right)\right|}\leq 2\left(N\frac{2M}{N}+\frac{M\sqrt{2}}{4}\ln N\right)\leq C_2M\ln N, \tag{18}$$

where $C_2$ is a positive constant.

By (6), (7), (16) and (18),

$$M\sum_{n=0}^{\infty}\left(R(n)-\lambda\right)^2r^{2n}+2M\left(2c+|\lambda|\right)^2N\frac{r^M}{1-r^M}+M\sum_{n=0}^{\infty}R(n)r^{2n}$$

$$\geq 2M\sum_{n=0}^{\infty} R(n)r^{2n} - 2\sqrt{\sum_{n=0}^{\infty} R(n)r^{2n}}\,|\lambda| C_2 M \ln N ,$$

that is,

$$\sum_{n=0}^{\infty}\left(R(n)-\lambda\right)^2 r^{2n} + 2\left(2c+|\lambda|\right)^2 N\frac{r^M}{1-r^M} \geq \sum_{n=0}^{\infty} R(n)r^{2n} - 2\sqrt{\sum_{n=0}^{\infty} R(n)r^{2n}}\,|\lambda| C_2 \ln N$$

. (19)

If, for example, $M = N^2$ then by (7), $r^M = \left(1-\frac{1}{N}\right)^M \leq \left(e^{-\frac{1}{N}}\right)^M = e^{-N} \to 0$ as $N \to \infty$, and so $\frac{r^M}{1-r^M} \to 0$ as $N \to \infty$. On the other hand, by (4) and (7),

$$\sum_{n=0}^{\infty} R(n)r^{2n} \geq \sum_{n=0}^{\infty} R^*(n)r^{2n} \geq \sum_{n=n_0}^{\infty} r^{2n} = r^{2n_0}\frac{1}{1-r^2} \geq \frac{1}{2}r^{2n_0}N \geq \frac{1}{3}N \quad (20)$$

for all sufficiently large $N$. Thus by (19) and (20),

$$\sum_{n=0}^{\infty}\left(R(n)-\lambda\right)^2 r^{2n} \geq \left(1-o(1)\right)\sum_{n=0}^{\infty} R(n)r^{2n} , \quad (21)$$

where $o(1)$ denotes a term which tends to $0$ as $N \to \infty$.

Let $\lambda$ be such a real number, for which $(2-\lambda)^2 < 2$ and $(4-\lambda)^2 < 4$ hold (that is, $2 < \lambda < 2+\sqrt{2}$). By (2) and (3),

$$R(n) = \begin{cases} 2R^*(n)-1 & \text{if } \frac{n}{2} \in \mathrm{A}, \\ 2R^*(n) & \text{otherwise.} \end{cases} \quad (22)$$

Therefore, by (4), there exists $\varepsilon > 0$ such that ($\varepsilon < 1$ and) $\left(R(n)-\lambda\right)^2 \leq R(n)-\varepsilon$ if $n \geq n_0$ ($n \in$ □ ) and $\frac{n}{2} \notin \mathrm{A}$, and so by (4), (6), (7), (20) and (22),

$$\sum_{n=0}^{\infty}\left(R(n)-\lambda\right)^2 r^{2n} \leq \sum_{n=0}^{n_0-1}\left(R(n)-\lambda\right)^2 + \sum_{n=n_0}^{\infty}\left(R(n)-\varepsilon\right)r^{2n} + \sum_{a\in\mathrm{A}}\left(R(2a)-\lambda\right)^2 r^{4a}$$

$$\leq C_3 + \sum_{n=n_0}^{\infty} R(n)r^{2n} - \varepsilon\sum_{n=n_0}^{\infty} r^{2n} + (2c+\lambda)^2\sum_{a\in\mathrm{A}} r^{2a}$$

$$\leq C_3 + \sum_{n=0}^{\infty} R(n)r^{2n} - \frac{\varepsilon}{4}\sum_{n=0}^{\infty} R(n)r^{2n} + \frac{\varepsilon}{4}\sum_{n=0}^{n_0-1} R(n)r^{2n} + (2c+\lambda)^2\sqrt{\sum_{n=0}^{\infty} R(n)r^{2n}}$$

$$= \left(1-\frac{\varepsilon}{4}+o(1)\right)\sum_{n=0}^{\infty} R(n)r^{2n} , \quad (23)$$

where $C_3$ is a positive constant.

By (21) and (23), we get

$$\left(1-o(1)\right)\sum_{n=0}^{\infty}R(n)r^{2n}\leq\left(1-\frac{\varepsilon}{4}+o(1)\right)\sum_{n=0}^{\infty}R(n)r^{2n},$$

i.e., $1-o(1)\leq 1-\frac{\varepsilon}{4}+o(1)$, and this contradiction proves the theorem.

**Reference**

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