Application of Fractional Calculus in Food Rheology

E. Vozáry, Gy. Csima, L. Csapó, F. Mohos

Szent István University,
Department of Physics and Control

Vozary.Eszter@etk.szie.hu

Keywords:

fractional calculus,
viscoelastic, food,
creep, recovery

Abstract. In fractional calculus the order (β) of differentiation or integration is not an integer number, generally β is a fractional number between 0 and 1. The root of fractional calculus goes back to the 18th century, and this calculus is intensively developing nowadays, too. Application of fractional calculus can be found in rheology, in electrical impedance spectroscopy, in physiological description. It is interesting, that the description of viscoelastic properties of biological material is much more accurate with fractional calculus than with ordinary differentiation and integration. In this work the creeping and recovery curves of a simple sweet, - gum candy –and bread slice, were approached with ordinary and fractional calculus. The rheological parameters of gum candy were determined. The fractional calculus gave better fit on the measured creep recovery curve points, than classical rheology models containing discrete elastic and viscous elements.

INTRODUCTION

In many cases the experimentally observed relaxation function exhibit a stretched (Kohlrausch) exponential decay $F(t) = F_0 \exp(-(t/\tau)^{\alpha})$, where F denotes the relaxing physical quantity (for example light intensity, stress relaxation in viscoelastic material, dielectric relaxation, etc.), t is the time, τ is a constant and $0 < \alpha < 1$ is a number (Schiessel et al., 1995). An appropriate tool to describe these relaxation processes is the fractional calculus (Süli, 2012).

The so called “Fractional Calculus” was born more than 300 years ago. In a letter dated September 30th, 1695 L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the nth-derivative of the linear function *f*(*x*) = *x*, $\frac{D^n x}{Dx^n}$ L'Hopital's posed the question to Leibniz, what would the result be if *n* = 1/2. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn."

Within the 20th century especially numerous applications and physical manifestations of fractional calculus have been found (Mainardi and Spada, 2011; Schiessel et al., 1995). While the physical meaning is difficult to grasp, the definitions themselves are no more rigorous than those of their integer order counterparts.

Within the 20th century especially numerous applications and physical manifestations of fractional calculus have been found (Mainardi and Spada, 2011;

Schiessel et al., 1995). While the physical meaning is difficult to grasp, the definitions themselves are no more rigorous than those of their integer order counterparts.

α order fractional integral according to Riemann-Liouville for a *f(t)* real or complex function can be given by the formula

$$(J^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt$$

where *t* is real variable, x>0, α>0 is a real number and $\Gamma(\alpha) = \int_0^{\infty} e^{-y} y^{\alpha-1} dy$ is the Gamma-function. There are several definitions for α order fractional derivative, practically as many as many mathematicians dealt with fractional calculus. A definition in which the Riemann-Liouville fractional integral is used can be given by the next expression:

$$(D^{\alpha})f(t) := \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\alpha)}\int_0^t \dfrac{f(\tau)d\tau}{(t-\tau)^{\alpha+1-m}} d(\tau)\right] \\ \dfrac{d^m}{dt^m} f(t) \end{cases}, \quad \begin{matrix} m-1<\alpha<m \\ \alpha = m \end{matrix}$$

where m-1<α<m and m is an integer number, α is a real number (Loverro, 2004).

In this work a stretched exponential function was fitted on creep and recovery curves of various food materials.

MATERIALS AND METHODS

The investigated materials: bread and gum candy were purchased in the local market. The creep-recovery test (CRT) curves were measured with a texture analyser TA-XT2 from Stable Micro System (Godalming, United Kingdom). The bread slices were pressed with a plexi cylinder of 36 mm diameter, and gum candies were pressed a metal cylinder of 75 mm diameter. The CRT test consists of four segments. The first segment is loading the sample with constant speed of measuring head until a pre-set force is reached. In the second segment the deformation is creeping under the constant force during a pre-set period. In the third segment (unloading) the probe is raised until the force on head becomes zero. In the fourth segment - in recovery - the relaxation of sample continues so, that the measuring head is raised when the relaxed sample reaches the probe. In our measurements the pre-set time was 60 s for both creeping and recovery period. The force in creeping period was 5 N and 2,5 N for gum candy and bread, respectively.

The creeping and recovery part of CRT are suitable for the determination of rheological parameters of sample material with model functions. The four-element Burgers model (Fig. 1) can describe both the creeping and recovery processes

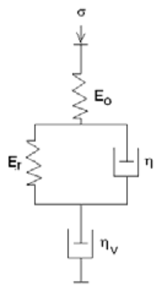

Figure 1.
The Burgers model. E_0 and E_r, represents the elastic modulus of the two spring elements and the η_r and η represents the viscosity of the two dashpots

The differential equation of four-element Burgers model (Sitkei, 1981) for σ stress and for ε strain in normal calculus

$$\frac{d^2\varepsilon}{dt^2}+\frac{1}{T_r}\frac{d\varepsilon}{dt}=\frac{1}{E_o}\left[\frac{d^2\sigma}{dt^2}+\left(\frac{E_o}{E_rT_r}+\frac{E_o}{\eta}+\frac{1}{T_r}\right)\frac{d\sigma}{dt}+\frac{E_o}{T_r\eta_v}\sigma\right]$$

and the solution of it is for creeping strain $\varepsilon(t)$ in time, t, when the stress is constant $\sigma=\sigma_0=\text{const}$:

$$\varepsilon(t)=\frac{\sigma_0}{E_0}+\frac{\sigma_0}{E_r}\left(1-e^{-\frac{t}{T_r}}\right)+\frac{\sigma_0}{\eta_v}t, \qquad (1)$$

where $T_r=\frac{\eta}{E_r}$, the retardation time and the solution of it for recovery strain $\varepsilon(t)$ in time, t, after t_1 time (time elapsed to beginning of recovery) when the stress becomes zero ($\sigma=0$):

$$\varepsilon(t)=\frac{\sigma_0}{E_r}\left(1-e^{-\frac{t_1}{T_r}}\right)e^{-\frac{t}{T_r}}+\frac{\sigma_0}{\eta_v}t_1 \qquad (2)$$

The differential equation of Burgers model in fractional calculus:

$$\frac{d^{\nu+1}\varepsilon}{dt^2}+\frac{1}{T_r}\frac{d^{\nu}\varepsilon}{dt}=\frac{1}{E_o}\left[\frac{d^{\nu+1}\sigma}{dt^2}+\left(\frac{E_o}{E_rT_r}+\frac{E_o}{\eta}+\frac{1}{T_r}\right)\frac{d^{\nu}\sigma}{dt}+\frac{E_o}{T_r\eta_v}\sigma\right]$$

and the solution is for creeping and recovery parts:

$$\varepsilon(t) = \frac{\sigma_0}{E_0} + \frac{\sigma_0}{E_r}\left(1 - e^{-\left(\frac{t}{T_r}\right)^{\beta}}\right) + \frac{\sigma_0}{\eta_v} t \qquad (3)$$

and

$$\varepsilon(t) = \frac{\sigma_0}{E_r}\left(1 - e^{-\left(\frac{t_1}{T_r}\right)^{\beta}}\right) e^{-\left(\frac{t}{T_r}\right)^{\beta}} + \frac{\sigma_0}{\eta_v} t_1 \qquad (4)$$

respectively, where $0 < \nu < 1$ and $0 < \beta < 1$

The (1) and (3) expressions were fitted on creeping curves and the (2) and (4) expressions on the recovery curves. The elastic modulus, viscosities and β parameter were determined.

RESULTS AND DISCUSSION

Typical creep-recovery curve of bread can be seen on Fig. 2. Similar CRT curve was measured on gum candies, too. The relative quick load part is followed by much longer creeping part. The creeping part of all CRT curves was approached with Burgers model of normal and fractional calculus, with equation (1) and (3). After the creeping part there is a relative quick unload segment which is followed by recovery curve. The recovery part of CRT curves was fitted by Burgers model both with normal and fractional calculus, with expressions (2) and (4).

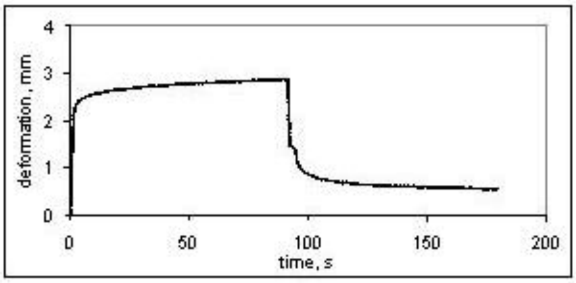

Figure 2.
A typical creep-recovery curve of a bread slice.

The result of curve fitting is demonstrated on Fig. 3. The stretched exponential function gave better approach of measured points especially in the beginning of both creeping and recovery part. The value of parameters from creeping period is very similar to the values from recovery period (Table 1.) for both investigated objects. Generally parameter values are lower from approaching the recovery part according to parameter values from creeping part.

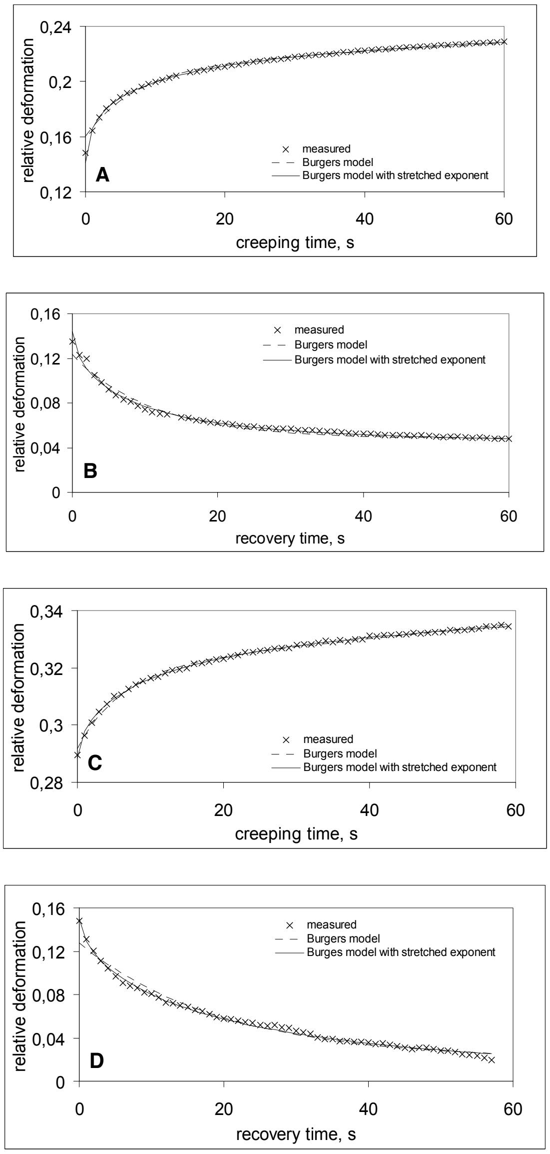

Figure 3
The fitting of creeping and recovery parts of CRT curve of bread (A,B) and of gum candy (C,D)

Table 1: The parameters of Burgers model from approaching the measured curves

Material	Calculus	part	E_0, kPa	E_r, kPa	η , MPas
bread	normal	creeping	16,2±0,8	52,2±3,2	0,39±0,02
		recovery		34,6±2,4	0,37±0,04
	fractional	creeping	18,4±1,0	31,8±3,5	0,25±0,06
		recovery		26,0±2,2	0,24±0,04
gum candy	normal	creeping	47,8±2,6	502,7±13,2	3,12±0,12
		recovery		127,2±8,2	2,57±0,19
	fractional	creeping	48,3±1,1	332,7±18,2	3,32±0,23
		recovery		88,2±7,5	1,72±0,10

Material	Calculus	part	η_V , MPas	β
bread	normal	creeping	8,73±0,72	
		recovery	5,11±0,64	
	fractional	creeping	16,39±0,98	0,521
		recovery	5,45±0,57	0,628
gum candy	normal	creeping	52,73±8,57	
		recovery	60,31±10,24	
	fractional	creeping	133,63±35,97	0,612
		recovery	258,73±29,76	0,625

It can be explained by the fact, that in the recovery period the stress is already zero, but in creeping period is about 2-3 kPa. Our earlier investigation showed, that both elastic moduli and viscosities of Burgers model for gum candy linearly increased, if the stress on the sample increased (Csima, 2015). This increase can be caused by structure changes under stress.

The lower parameter values for bread according to parameter values for gum candy can be explained with the lower elasticity and hardness of bread compared to gum candy.

CONCLUSIONS

The approach of creep and recovery part of CRT curves was proved more precise with fractional calculus than with normal calculus. It seems that the stretched exponential function better describes especially the quick processes of both creeping and recovery curves.

REFERENCES

Csima Gy. (2015) Zselatin alapú édesipari termék reológiájának jellemzése, PhD. disszertáció, Budapesti Corvinus Egyetem.

Loverro, A. (2004): Fractional Calculus: History, Definitions and Applications for Engineer http://www3.nd.edu/’’msen/Teaching/UnderRes/Frac.Calc.pdf

Mainardi, F. and Spada, G. (2011): Creep, relaxation and viscosity properties for basic fractional models in rheology.

The Europian Physical Journal Special Topics, 193:133-160.

Schiessel, H., Metzler, R., Blumen, A., Nonnenmacher, T.F. (1995): Generalized viscoelastic models: their fractional equations with solutions. *Journal of Physics A: Mathematical and General*, 28:6567-6584.

SITKEI, GY. (1981): *A mezőgazdasági anyagok mechanikája*. Akadémiai Kiadó, Budapest

Süli B. M. (2012): Mi legyen egy függvény $\sqrt{2}$ -edik deriváltja?. Szakdolgozat, Eötvös Loránd Tudományegyetem, Természettudományi kar, Analízis Tanszék