Asymmetric general Choquet integrals

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Abstract: A notion of a generated chain variation of a set function m with values in \([-1, 1]\) is proposed. The space \(B_{gV}\) of set functions of bounded g-chain variation is introduced and properties of set functions from \(B_{gV}\) are discussed. A general Choquet integral of bounded \(A\)-measurable function is defined with respect to a set function \(m \in B_{gV}\). A constructive method for obtaining this asymmetric integral is considered. A general fuzzy integral of bounded g-variation, comonotone \(\oplus\)-additive and positive \(\odot\)-homogenous is represented by a general Choquet integral. The representation of a general Choquet integral in terms of a pseudo Lebesque-Stiltjes integral is obtained.

Keywords: symmetric pseudo-operations, non-monotonic set function, general fuzzy integral, asymmetric Choquet integral

1 Introduction

The Choquet integral is often used in economics, pattern recognition and decision analysis as nonlinear aggregation tool [4, 5, 6, 20, 21, 23, 24]. Most of the studies of non-additive set functions and integrals have been focused to the case when their values are in non-negative interval (fuzzy measures), e.g., \([0, 1]\). A fuzzy measure \(m : \mathcal{A} \rightarrow [0, 1]\) (or \([0, \infty]\), \(m(\emptyset) = 0\)) is a non-decreasing set function, defined on \(\sigma\)-algebra \(\mathcal{A}\). Integrals can be viewed as an extension of underlining measures, see [9, 10].

Choquet integral (introduced in [3]) of \(\mathcal{A}\)-measurable non-negative function \(f\) with respect to a fuzzy measure \(m : \mathcal{A} \rightarrow [0, \infty]\) is defined by

\[
C_m(f) = \int_0^\infty m\{x|f(x) \geq t\} dt.
\]

The main properties of the Choquet integral are monotonicity and comonotone additivity, see [4, 18]. For a finite fuzzy measure \(m\) and \(\mathcal{A}\)-measurable \(f : X \rightarrow \mathbb{R}, f^+ = f \vee 0, f^- = f \wedge 0,\)
\( f^- = (-f) \vee 0 \) we have
\[
C_m(f) = C_m(f^+) - C_{\overline{m}}(f^-),
\]
where \( \overline{m} \) is the conjugate set function of a fuzzy measure \( m \), given by \( \overline{m}(E) = m(X) - m(E^c) \), for \( E \in \mathcal{A} \), where \( E^c = X \setminus E \). The last integral is known under the name asymmetric Choquet integral. In [16] it has been shown that this integral is well defined on the class of bounded \( \mathcal{A} \)-measurable functions with respect to all real-valued set functions, \( m : \mathcal{A} \to \mathbb{R} \) of bounded chain variation, such that \( m(\emptyset) = 0 \), even if they are non-monotonic. The asymmetric Choquet integral is linear with respect to \( m \), hence (see [16, 18])
\[
C_m(f) = C_{m_1}(f) - C_{m_2}(f).
\]

Fuzzy integrals corresponding to an appropriate couple \((\oplus, \odot)\) of pseudo-operations have been studied in [12, 13, 17, 18, 19, 25]. Symmetric pseudo-operations are introduced in [6, 7]. A construction of general fuzzy integral has been studied in [2, 10, 25]. As a special type of such integral, the Choquet-like integral, introduced in [12], is defined with respect to pseudo-operations with a generator. It can be viewed as a transformation of the Choquet integral. The Choquet-like integral related to some non-decreasing function \( g : [0, 1] \to [0, \infty], g(0) = 0 \), defined for a non-negative \( \mathcal{A} \)-measurable function \( f \) and a fuzzy measure \( m \), is given by
\[
C_g^m(f) = g^{-1}(C_{\text{com}}(g \circ f)) \tag{1}
\]
This integral is also defined for a real-valued function \( f \), if for \( g \) is taken its odd extension to the whole real line [12, 13], and we shall call it a general Choquet integral.

The aim of this paper is to present a general Choquet integral defined with respect to set functions of bounded \( g \)-chain variation. As we shall see, this integral is of bounded \( g \)-variation asymmetric, comonotone \( \oplus \)-additive and positively \( \odot \)-homogenous.

The paper is organized as follows. Section 2 is devoted to preliminary notions and definitions of symmetric pseudo-operations. In Section 3 we introduce a \( g \)-chain variation of set functions and we consider the space of set functions of bounded \( g \)-chain variation \( BgV \). In Section 4 we introduce the notion of a signed \( \oplus_g \)-measure. A pseudo-difference representation of a signed \( \oplus_g \)-measure is obtained. In Section 5 we introduce a general fuzzy integral defined with respect to \( m \in BgV \). We consider its relation with the asymmetric general Choquet integral, i.e., Choquet-like integral (defined by (1), w.r.t. \( m \in BgV \)) and present its representation in the term of a pseudo Lebesgue-Stieltjes integral. As a consequence, in the case of an underlining signed \( \oplus_g \)-measure this integral reduces to a pseudo Lebesgue integral.

### 2 Symmetric pseudo-operations

We recall definitions of a t-conorm and pseudo-operations according to [6, 7, 9, 10].

**Definition 1** A triangular conorm (t-conorm) is a commutative, associative, non-decreasing function \( S : [0, 1]^2 \to [0, 1] \), with neutral element 0.
**Definition 2** An additive generator \( s : [0, 1] \rightarrow [0, \infty] \) of a t-conorm \( S \) (if it exists) is left continuous at 1, increasing function, such that \( s(0) = 0 \), and for all \( (x, y) \in [0, 1]^2 \) we have

\[
S(x, y) = s^{-1}(s(x) + s(y)) ,
\]

\[
s(x) + s(y) \in \text{Ran}(s) \cup [s(1), \infty],
\]

where \( s^{-1} \) is a pseudo-inverse function of \( s \) (see [9]).

**Definition 3** Let \( S : [0, 1]^2 \rightarrow [0, 1] \) be a continuous triangular conorm. Pseudo-addition \( \oplus_s : [-1, 1]^2 \rightarrow [-1, 1] \), is defined by

\[
x \oplus_s y = \begin{cases} 
S(x, y), & (x, y) \in [0, 1]^2 \\
-S(|x|, |y|), & (x, y) \in [-1, 0]^2 \\
a, & (x, y) \in [0, 1] \times (-1, 0], x \geq |y| \\
b, & (x, y) \in [0, 1] \times [-1, 0], x \leq |y| \\
1 \text{ or } -1, & (x, y) \in \{(1, -1), (-1, 1)\} \\
y \oplus_s x, & \text{else},
\end{cases}
\]

where \( a = \inf\{z \mid S(-y, z) \geq x\} \) and \( b = -\inf\{z \mid S(x, z) \geq -y\} \).

The binary operation \( \oplus_s \) is commutative, monotone, with neutral element 0. If it is associative, e.g., if \( S \) is a strict t-conorm, \( \oplus_s \) can be extended to \( n \)-ary operation. Then for all \( n \)-tuple \( (x_1, x_2, \ldots, x_n) \in [-1, 1]^n \) we define:

\[
\bigoplus_{i=1}^{n} x_i = \left( \bigoplus_{i=1}^{n-1} x_i \right) \oplus_s x_n. \tag{2}
\]

**Definition 4** Let \( S \) be a continuous t-conorm. The pseudo-difference associated to t-conorm \( S \) is given by:

\[
x \ominus_s y = x \oplus_s (-y) \tag{3}
\]

for all \( (x, y) \in [-1, 1]^2 \setminus \{(1,1), (-1, -1)\} \). By the convention \( 1 \ominus_s 1 = a, a \in \{\pm 1, 0\} \).

**Example 1** For all \( (x, y) \in [-1, 1]^2 \setminus \{(1,1), (-1, -1)\} \) and for maximum \( \vee \), Yager t-conorm \( S_p^\triangledown \) and Hamacher t-conorm (Einstein sum) \( S_p^\Delta \) (see [10]), we have, respectively:

(i) \( x \ominus_s y = \text{sign}(x-y)(|x| \vee |y|) \):

(ii) For \( p = 2k - 1 \),

\[
x \ominus_s S_p^\triangledown y = \begin{cases} 
-1, & x^p - y^p < -1, \\
\frac{x^p - y^p}{y^p}, & -1 \leq x^p - y^p \leq 1, \\
1, & x^p - y^p > 1;
\end{cases}
\]

(iii) \( x \ominus_s S_p^\Delta y = \frac{x-y}{1-sy} \).
Let $S$ be a strict t-conorm with an additive generator $s : [0, 1] \to [0, \infty]$. Let $g : [-1, 1] \to [-\infty, \infty]$ be defined by:

$$g(x) = \begin{cases} 
    s(x), & x \geq 0 \\
    -s(-x), & x < 0 
\end{cases} \quad (4)$$

The function $g$ is the symmetric extension of $s$, so it is a strictly increasing function.

A pseudo-addition $\oplus$ can be transformed to a binary operation $U$ on $[0, 1]$, i.e., to a generated uninorm. The results contained in the following proposition have been shown in [6, 7, 9].

**Proposition 1** Let $S$ be a strict t-conorm with an additive generator $s$, pseudo-addition $\oplus$ and function $g$ defined by (4), then:

(i) for all $x, y \in [0, 1]$

$$x \ominus y = g^{-1}(g(x) - g(y));$$

(ii) for all $x, y \in [-1, 1]$

$$x \oplus y = g^{-1}(g(x) + g(y)); \quad (5)$$

(iii) for all $z, t \in [0, 1]$

$$U(z, t) = u^{-1}(u(z) + u(t)),$$

where $u : [0, 1] \to [-\infty, \infty]$, is given by $u(x) = g(2x - 1)$, with the convention $\infty - \infty \in \{\infty, -\infty\}$.

It is clear that (i) holds for all $(x, y) \in [-1, 1]^2 \setminus \{(1, 1), (-1, -1)\}$. It is shown in [7] that $([-1, 1], \oplus)$ is an Abelian group.

It is a well known fact that a pseudo-multiplication $\odot : [-1, 1] \to [-1, 1]$, which is distributive with respect to $\oplus$, can be defined using the additive generator of pseudo-addition $\oplus$, i.e., for $g : [-1, 1] \to [-\infty, \infty]$, $\odot$ is defined by:

$$x \odot y = g^{-1}(g(x)g(y)), \quad (6)$$

for all $(x, y) \in [-1, 1]^2$. The pseudo-multiplication defined in this manner is commutative, associative with neutral element $e_\odot \in [0, 1]$ and distributive with respect to pseudo-addition $\oplus$.

**Example 2** Let $\oplus_{sp}$ be the pseudo-addition induced by the probabilistic sum $S_p : [0, 1]^n \to [0, 1]$, defined by

$$S_p(x_1, x_2, \ldots, x_n) = 1 - \prod_{i=1}^n (1 - x_i).$$

The additive generator $g$ of $\oplus_{sp}$ is defined by:

$$g(x) = \begin{cases} 
    -\ln(1-x), & x \geq 0 \\
    -\ln(1)+x, & x < 0 
\end{cases}$$
Let $\odot$ be given by: \( x \odot y = g^{-1}(g(x)g(y)) \), for all \( x, y \in [-1, 1] \), i.e.,
\[
x \odot y = \text{sign}(x \cdot y) \left( 1 - e^{-\ln(1-|x|)\ln(1-|y|)} \right).
\]

For all \( x \in [-1, 1]\setminus\{0\} \) we have:
\[
x \odot e_\odot = x \quad x \odot x^{-1} = e_\odot,
\]
where the neutral element is given by \( e_\odot = 1 - \frac{1}{e} \), and an inverse element, for \( x \in [-1, 1]\setminus\{0\} \) is given by \( x^{-1} = \text{sign}(x) \left( 1 - e^{-\frac{1}{\ln(1-|x|)}} \right) \). Hence, (\( -1, 1\setminus\{0\}, \odot \)) is an Abelian group.

The following result was shown in [15].

**Proposition 2** Let \( S \) be a strict \( t \)-conorm, pseudo-addition \( \oplus \) with the generating function \( g \) given by (4), and pseudo-multiplication \( \odot \) is defined by (6). Then we have:

(i) \( [-1, 1]; \oplus, \odot \) is a field isomorphic to \((\mathbb{R}, +, \cdot)\)

(ii) The pseudo-multiplication has the next form
\[
x \odot y = \text{sign}(x \cdot y) U^\odot(|x|, |y|),
\]
where the uninorm \( U^\odot : [0, 1]^2 \rightarrow [0, 1] \) is defined by \( U^\odot(x, y) = s^{-1}(s(x)s(y)) \) for all \( x, y \in [0, 1] \), with the convention:

(a) in the case \( \infty \cdot 0 = 0 \), \( U^\odot \) is conjunctive,

(b) in the case \( \infty \cdot 0 = \infty \), \( U^\odot \) is a disjunctive uninorm.

It is clear now, that the couple of symmetric pseudo-operations \((\oplus, \odot)\) can be expressed in terms of a couple of uninorms, or as it is usual by (5) and (6).

### 3 Space \( B_{gV} \)

According to [16, 18], the chain variation of a real valued set function \( m : \mathcal{A} \rightarrow \mathbb{R} \), \( m(\emptyset) = 0 \), for all \( E \in \mathcal{A} \), is defined by
\[
|m|(E) = \sup \left\{ \sum_{i=1}^{n} |m(E_i) - m(E_{i-1})| \mid \emptyset = E_0 \subset \ldots \subset E_n = E, \quad E_i \in \mathcal{A}, i = 1, \ldots, n \right\},
\]
where supremum is taken with respect to all finite chains from \( \emptyset \) to \( E \). The chain variation \( |m| \) of a real-valued set function \( m \) is positive, monotone, set function, \( |m|(\emptyset) = 0 \) and \( |m|(E) \leq |m|(E) \) for all \( E \in \mathcal{A} \). We say that a real-valued set function \( m, m(\emptyset) = 0 \), is of bounded chain variation if \( |m|(X) < \infty \), and we denote by \( BV \) the set of all set functions with the bounded chain variation, vanishing at the empty set. We refer [1, 16, 18] for an exhaustive overview of properties and results related to \( BV \). It is proven in [1, 18] that a real-valued set function \( m \) belongs to \( BV \) if it can be represented as difference of two monotone set functions \( v_1 \) and \( v_2 \).
Definition 5 [15] For a given function \( g : [-1, 1] \to [-\infty, \infty] \), defined by (4), g-chain variation \( |m|_g \) of a set function \( m : \mathcal{A} \to [-1, 1], m(\emptyset) = 0 \), is defined by

\[
|m|_g(E) = g^{-1} \left( \sup \left\{ \sum_{i=1}^{n} |g(m(E_i)) - g(m(E_{i-1}))| \right\} \right),
\]

for all \( E \in \mathcal{A} \) and supremum is taken with respect to all finite chains.

Using the fact that \( g \) is an odd function, we easily obtain the following result.

Proposition 3 Let \( m : \mathcal{A} \to [-1, 1] \) be a set function, \( m(\emptyset) = 0 \), then g-chain variation has the following properties:

(i) \( 0 \leq |m|_g(E) \leq 1, \quad E \in \mathcal{A} \).

(ii) \( |m|_g(\emptyset) = 0 \).

(iii) \( |m(E)| \leq |m|_g(E), \quad E \in \mathcal{A} \).

(iv) \( |m|_g \) is a monotone set function, i.e.,

\[
|m|_g(E) \leq |m|_g(F),
\]

for all \( E \subset F, E, F \in \mathcal{A} \).

(iv) If \( m : \mathcal{A} \to [0, 1] \) is a monotone set function, then

\[
|m|_g(E) = m(E) \quad \text{for all} \quad E \in \mathcal{A}.
\]

We say that a set function \( m : \mathcal{A} \to [-1, 1], m(\emptyset) = 0 \), is of bounded g-chain variation if \( |m|_g(X) < 1 \), and we denote by \( BgV \) the family of such set functions.

Proposition 4 Let \( m_1, m_2 \in BgV \). Then

\[
|m_1 \oplus_s m_2|_g(X) \leq |m_1|_g(X) \oplus_s |m_2|_g(X).
\]

Proof: We will use the next notation

\[
L = \{ \emptyset = E_0 \subset E_1 \subset \ldots \subset E_n = F, \quad E_i \in \mathcal{A}, i = 1, \ldots, n \}.
\]

We denote by \( \mathcal{C}_F \) all finite chains from \( \emptyset \) to \( F \). We have

\[
|m_1 \oplus_s m_2|_g(X) = g^{-1} \left( \sup_{L \in \mathcal{C}_X} \left\{ \sum_{i=1}^{n} \left| g((m_1 \oplus_s m_2)(E_i)) - g((m_1 \oplus_s m_2)(E_{i-1})) \right| \right\} \right)\\
= g^{-1} \left( \sup_{L \in \mathcal{C}_X} \left\{ \sum_{i=1}^{n} |g \circ m_1(E_i) + g \circ m_2(E_i) \right\} \right).
\]
\[-g \circ m_1(E_{i-1}) - g \circ m_2(E_{i-1})\}
\leq g^{-1}\left(\sup_{L \in \mathcal{C}} \left\{ \sum_{i=1}^{n} |g \circ m_1(E_i) - g \circ m_1(E_{i-1})|\right\}\right)
+ \sum_{i=1}^{n} |g \circ m_2(E_i) - g \circ m_2(E_{i-1})|\right\}\right)
\leq g^{-1}\left(g\left(\sup_{L \in \mathcal{C}} \left\{ \sum_{i=1}^{n} |g \circ m_1(E_i) - g \circ m_1(E_{i-1})|\right\}\right)\right)
+ g\left(g^{-1}\left(\sup_{L \in \mathcal{C}} \left\{ \sum_{i=1}^{n} |g \circ m_2(E_i) - g \circ m_2(E_{i-1})|\right\}\right)\right)
= |m_1|_g(X) \oplus_S |m_2|_g(X).\]

\[\square\]

**Proposition 5** [15] A set function \(m : \mathcal{A} \rightarrow [-1,1]\), \(m(\emptyset) = 0\), belongs to \(Bgv\) if and only if it can be represented as follows
\[
m = m_1 \oplus_S m_2,
\]
where \(m_1, m_2 : \mathcal{A} \rightarrow [0,1]\) are two fuzzy measures.

**Proof:** We have that \(m \in Bgv\) if and only if \(g \circ m \in BV\). By Theorem 3.10 from [18], there exist two fuzzy measures \(\hat{m}_1\) and \(\hat{m}_2\) such that \(g \circ m = \hat{m}_1 - \hat{m}_2\). Taking \(m_1 = g^{-1} \circ \hat{m}_1\) and \(m_2 = g^{-1} \circ \hat{m}_2\) we obtain the claim. \(\square\)

## 4 Signed \(\oplus_S\)-measures

In this section we consider a set function \(m : \mathcal{A} \rightarrow [-1,1]\). We will define \(\sigma\)-\(\oplus_S\)-additivity of a set function \(m\) in the following manner. Let \(S\) be a strict t-conorm and \(\oplus_S\) a pseudoaddition with an additive generator \(g : [-1,1] \rightarrow [\infty, \infty]\). First, we define the notion of a convergent \(\oplus_S\)-series \(\bigoplus_{i=1}^{n} a_i\). We have the following situations:

(i) An expression \(\bigoplus_{i=1}^{n} a_i\) is unambiguously defined if \(a_i \geq 0\) for all \(i = 1, 2, \ldots\). Then \(\big\{ \bigoplus_{i=1}^{n} a_i \big\}_{n \in \mathbb{N}}\) is a monotone increasing sequence of reals from the interval \([0,1]\), hence
\[
\bigoplus_{i=1}^{\infty} a_i := \lim_{n \to \infty} \bigoplus_{i=1}^{n} a_i,
\]
i.e., the sum of \(\oplus_S\)-series is equal to a number from the interval \([0,1]\) and we say that \(\oplus_S\)-series is convergent, otherwise it diverges to 1.

(ii) In the case when \(a_i \leq 0\), for all \(i = 1, 2, \ldots\) we have the similar situation as in (i), i.e., the sum of \(\oplus_S\)-series is a number from the interval \([-1,0]\), otherwise it diverges to
−1.

(iii) For $a_i \in [-1, 1]$, $i = 1, 2, \ldots$, analogously as in the previous situations, we take (7), i.e., the classical limit value of the sequence of reals $\left\{ \sum_{i=1}^{n} a_i \right\}_{n \in \mathbb{N}}$, if it exists, i.e., if it is a number from the interval $[-1, 1]$.

We introduce the notion of $\sigma$-$\oplus_s$-additivity as follows. A distorted signed measure $\mu$ transformed by $g^{-1}$, i.e., any real valued signed fuzzy measure $m = g^{-1} \circ \mu$ is $\sigma$-$\oplus_s$-additive, if $g$ is an additive generator of pseudo-addition $\oplus_s$ and $\mu : \mathcal{A} \rightarrow [-\infty, \infty]$ is an arbitrary signed measure.

**Definition 6** A set function $m : \mathcal{A} \rightarrow [-1, 1]$ is a signed $\oplus_s$-measure if there exists a signed measure $\mu : \mathcal{A} \rightarrow [-\infty, \infty]$ such that:

$$
m E_i g^{-1}\left( \sum_{i=1}^{\infty} \mu E_i \right)
$$

is fulfilled for any sequence $\{E_i\}_{i \in \mathbb{N}}$, $E_i \in \mathcal{A}$, satisfying $E_k \cap E_j = \emptyset$ for $k \neq j$, where the series on the right side is either convergent or divergent to $+\infty$ or $-\infty$.

Obviously, we have $m(\emptyset) = 0$ and $m$ takes on at most one of the values from $\{-1, 1\}$.

**Proposition 6** Let $m : \mathcal{A} \rightarrow [-1, 1]$ be a signed $\oplus_s$-measure. Then there exist unique $S$-measures $m_1$ and $m_2$ such that

$$
m = m_1 \oplus_s m_2.
$$

**Proof.** According to the classical Jordan’s theorem of representation of a signed measure (see [8]), we have $\mu = \mu^+ - \mu^-$, where $\mu^+$ and $\mu^-$ are measures. By Definition 6, for all $E \in \mathcal{A}$ we have

$$
m E = g^{-1}(\mu E) \\
= g^{-1}(\mu^+(E) - \mu^-(E)) \\
= g^{-1}(g(g^{-1} \circ \mu^+(E)) - g(g^{-1} \circ \mu^-(E))) \\
= m_1(E) \oplus_s m_2(E).
$$

**Example 3** Let $\mu : \mathcal{A} \rightarrow [-\infty, \infty]$ be a signed measure and let $m$ be a set function defined on $\sigma$-algebra $\mathcal{A}$, $m : \mathcal{A} \rightarrow [-1, 1]$ as follows:

$$
m E = \text{sign}(\mu E) \left( 1 - e^{-|\mu E|} \right).
$$

The set function $m$ is a signed $\oplus_{sp}$-measure.

**Remark 1** Let $m : \mathcal{A} \rightarrow [-1, 1]$ be a set function such that $m \in BgV$. Then there exist $m_1$ and $m_2$ such that $m = m_1 \oplus_s m_2$. If the fuzzy measures $m_1$ and $m_2$ are $S$-measures, then $m$ is a signed $\oplus_S$-measure.
5 A general Choquet integral

Let \((X, \mathcal{A})\) be a measurable space, and \(\mathcal{F}^+\) and \(\mathcal{F}\) classes of \(\mathcal{A}\)-measurable functions given by

\[
\mathcal{F}^+ = \{ f \mid f : X \rightarrow [0, 1], \sup_{x \in X} f(x) < 1 \}, \\
\mathcal{F} = \{ f \mid f : X \rightarrow [-1, 1], \sup_{x \in X} |f(x)| < 1 \}.
\]

Let the operation \(\ominus\) be given by Definition 4. For a set function \(m : \mathcal{A} \rightarrow [-1, 1]\), \(m(\emptyset) = 0\), we define a pseudo conjugate set function \(m^{\ominus} : \mathcal{A} \rightarrow [-1, 1]\) by:

\[
m^{\ominus}(E) = m(X) \ominus m^c(E),
\]

for all \(E \in \mathcal{A}\), where \(E^c = X \setminus E\).

**Proposition 7** [15] We have

(i) \(f = f^+ \ominus f^-\), for any \(f \in \mathcal{F}\), where \(f^+, f^- \in \mathcal{F}^+\), \(f^+ = f \vee 0\) and \(f^- = (-f) \vee 0\).

(ii) \(m\) is monotone if and only if \(m^{\ominus}\) is monotone.

(iii) Let \(m_1, m_2 : \mathcal{A} \rightarrow [-1, 1]\) such that \(m_1(X) = m_2(X)\). Then

\[
m_1 \leq m_2 \text{ if and only if } m_1^{\ominus} \geq m_2^{\ominus}.
\]

In the sequel, \(\oplus\) and \(\otimes\) will denote associative pseudo-operations, defined by (5) and (6), respectively, and \(\ominus\) the corresponding pseudo-difference. The measurable functions \(f\) and \(h\) on \(X\) are called comonotone [4] if they are measurable with respect to the same chain \(C\) in \(\mathcal{A}\). Equivalently, comonotonicity of functions \(f\) and \(h\) can be expressed as follows: \(f(x) < f(x_1) \Rightarrow h(x) \leq h(x_1)\) for all \(x, x_1 \in X\).

**Definition 7** Let \(I : \mathcal{F} \rightarrow [-1, 1]\) be a functional. We say that

(i) \(I\) is monotone if for all \(f, h \in \mathcal{F}\)

\[
f \leq h \Rightarrow I(f) \leq I(h),
\]

(ii) \(I\) is comonotone \(\oplus\)-additive if

\[
I(f \oplus h) = I(f) \oplus I(h)
\]

for all comonotone \(f\) and \(h\) from \(\mathcal{F}\),

(iii) \(I\) is positively \(\otimes\)-homogenous if

\[
I(a \otimes f) = a \otimes I(f)
\]

for all \(a \in [0, 1], f \in \mathcal{F}\).
is of bounded g-variation if \( G(I) < 1 \), where a g-variation \( G(I) \) of \( I \) is defined by
\[
G(I) = g^{-1} \left( \sup \left\{ \sum_{i=1}^{n} |g(I(h_i)) - g(I(h_{i-1}))| \mid 0 = h_0 \leq \ldots \leq h_n = e1_X, \ h_i \in \mathcal{F} \right\} \right).
\]

**Remark 2** Obviously, if \( I : \mathcal{F} \to [-1, 1] \) is a monotone functional, then g-variation of \( I \) is given by \( G(I) = I(e1_X) \).

Let \( m \in BgV \) and let \( s \in \mathcal{F} \) be a simple function with \( Ran(s) = \{ s_1, s_2, \ldots, s_n \} \). We define
\[
I_m(s) = s_1 \odot m(E_1) \odot \bigoplus_{i=2}^{n} (s_i \odot s_{i-1}) \odot m(E_i),
\]
where \(-1 < s_1 \leq s_2 \leq \ldots \leq s_n < 1 \) and \( E_i = \{ x \in X \mid s(x) \geq s_i \} \).

**Proposition 8** [15] Let \( I_m \) be defined by (8). For all simple functions from \( \mathcal{F} \), and for all \( m \in BgV \) we have:

(i) \( I_m \) satisfies the properties (ii) and (iii) given in Definition 7.
(ii) \( I_m(s) = I_m(s^+) \odot I_m^+(s^-) \).
(iii) \( I_m(s) = I_{m_1}(s) \odot I_{m_2}(s) \), where \( m_1 \) and \( m_2 \) are given by Proposition 5.
(iv) \( I_m(a \cdot 1_E) = \begin{cases} a \odot m(E) & a \in [0, 1] \\ a \odot m^+(E) & a \in [-1, 0] \end{cases} \).

We consider now a general fuzzy integral. First we define a general fuzzy integral with respect to a monotone, non-negative function \( m \in BgV \) and then with respect to an arbitrary \( m \) from \( BgV \).

**Definition 8** A general fuzzy integral \( I_m : \mathcal{F} \to [-1, 1] \) is defined by:

(i) For a fuzzy measure \( m \) from \( BgV \)
\[
I_m(f) = \sup_{s \in \mathcal{F}^+, s \leq f^+} I_m(s) \oplus \inf_{-s' \in \mathcal{F}^-, f' \leq f^-} I_m(s').
\]
(ii) For \( m \in BgV \)
\[
I_m(f) = I_{m_1}(f) \odot I_{m_2}(f),
\]
where \( m_1 \) and \( m_2 \) are given by Proposition 5.

A general fuzzy integral \( I_m : \mathcal{F} \to [-1, 1] \) with respect to a fuzzy measure is monotone. \( I_m \) is asymmetric, i.e.,
\[
I_m(-f) = -I_{m^+}(f),
\]
for all \( f \in \mathcal{F} \).
Proposition 9 Let $I_m : \mathcal{F} \to [-1, 1]$ be a general fuzzy integral with respect to $m \in \text{BgV}$. We have:

(i) $I_m$ is of bounded $g$-variation.

(ii) $I_m$ satisfies the properties (ii) and (iii) given in Definition 7.

(iii) $I_m(f) = I_m(f^+) \ominus I_m(f^-)$, for all $f \in \mathcal{F}$.

Proof. (i) Let $m \in \text{BgV}$, by Proposition 5, $m = m_1 \ominus m_2$, where $m_1$ and $m_2$ are fuzzy measures from $\text{BgV}$. $I_{m_1}, I_{m_2} : \mathcal{F} \to [-1, 1]$ are monotone functionals. By definition of $g$-variation we have $G(-1) = G(1)$ and

$G(I_m) = G(I_m \ominus m_2) \leq G(I_{m_1}) \ominus G(I_{m_2}) = I_{m_1}(e1_{X}) \ominus I_{m_2}(e1_{X}) = m_1(X) \ominus m_2(X) < 1.$

We obtain (ii) and (iii) by (8), (9), (10) and Proposition 8.

Based on the above consideration and results proven in [2, 4, 15, 16, 18] we have the next propositions.

Proposition 10 Let $I_m : \mathcal{F} \to [-1, 1]$ be a general fuzzy integral with respect to $m \in \text{BgV}$. Then

$I_m(f) = C_m^g(f) = g^{-1}(C_{\text{gcm}}(g \circ f)),$

where $C_m^g$ is a general Choquet integral.

Proposition 11 Let $I_m : \mathcal{F} \to [-1, 1]$ be a general fuzzy integral w.r.t. $m \in \text{BgV}$. Then

$I_m(f) = g^{-1}\left(LS \int_{[-\infty, \infty]} g(t)d(g \circ F)(t)\right),$ 

where the integral on the right-hand side is a pseudo Lebesgue-Stieltjes integral.

Proof. Let $F : [-1, 1] \to [-1, 1]$ be a function of bounded totally $g$-variation, i.e.,

$g^{-1}\left(\sup_{i=1}^{n} \left| g(F(t_i)) - g(F(t_{i-1})) \right| : -1 \leq t_1 \leq \ldots \leq t_n \leq 1, i = 1, \ldots, n \right) < 1.$

Then there exist two non-decreasing functions $F^+$ and $F^-$ such that $F = F^+ \ominus F^-$ and a signed $\oplus$-measure on a $\sigma$-algebra of Borel subsets of $[-1, 1]$, induced by $F$.

Let $I_m$ be a general fuzzy integral with respect to $m \in \text{BgV}$. For $f \in \mathcal{F}$, let $F$ be defined by

$F(t) = -m\{x \in X | f(x) \geq t\}, \quad t \in [-1, 1].$

$F$ is of bounded totally $g$-variation (11). $f \in \mathcal{F}$ is bounded, therefore $g \circ f$ is bounded, $I_m(f) = C_m^g(f)$, and according to [16] (Appendix) we have the claim.

Corollary 1 Let $I_m : \mathcal{F} \to [-1, 1]$ be a general fuzzy integral with respect to a signed $\oplus$-measure $m, m \in \text{BgV}$. Then

$I_m(f) = g^{-1}\left(\int g \circ f d\mu\right),$

where integral on the right-hand side is $g$-integral, see [17, 18].
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